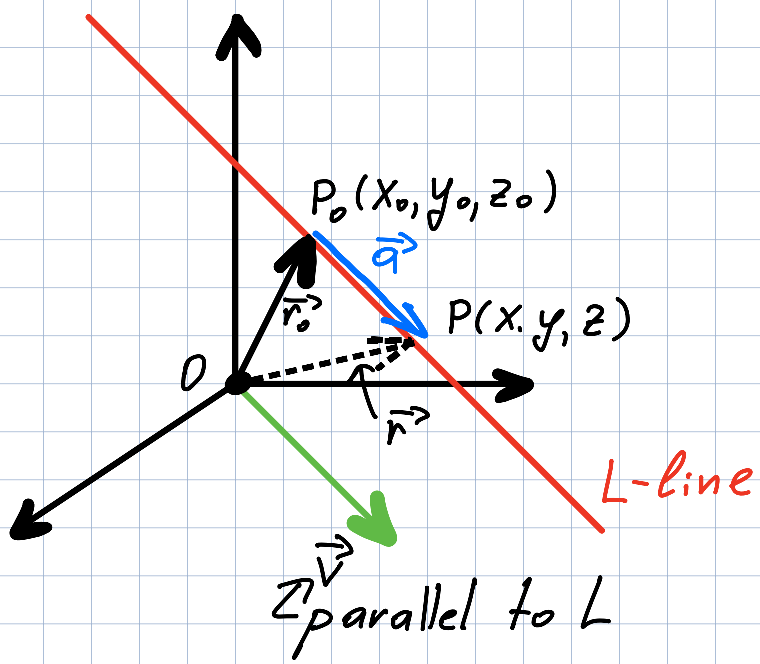
Lines



$P_0(x_0, y_0, z_0)$ is a fixed point on L

$P(x, y, z)$ – some (any) point on L

$\vec{r_0}$ is the position vector of P_0

$(\vec{r_0} = \overrightarrow{OP_0})$

$\vec{r}$ - position vector of P

Then $\vec{r} = \vec{r_0} + \vec{a} \Rightarrow$

$(\vec{a} \parallel \vec{v} \Rightarrow \vec{a} = t\vec{v})$ $\boxed{\vec{r} = \vec{r_0} + t\vec{v}}$

– vector equation of L

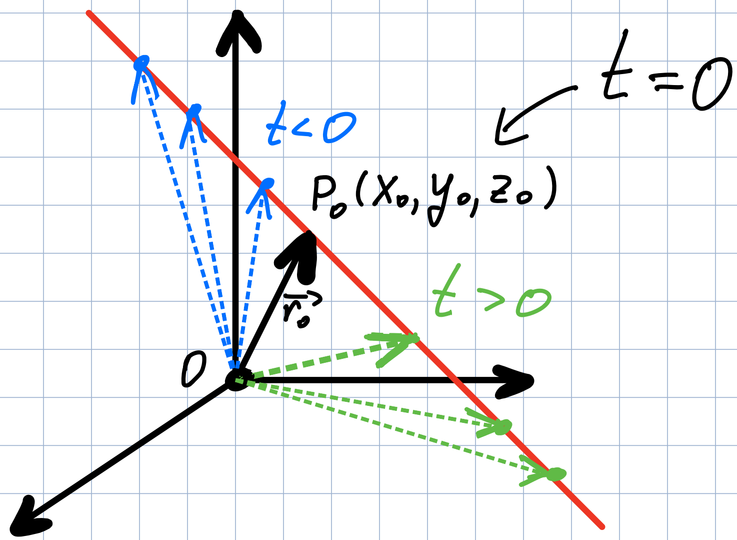
Let $\vec{v} = \langle a, b, c \rangle$ then in components:

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

$t \in \mathbb{R}$ parameter

$$\boxed{\begin{cases} x = x_0 + ta \\ y = y_0 + tb \\ z = z_0 + tc \end{cases}}$$

parametric equations of the line L through point $P_0(x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = \langle a, b, c \rangle$



Ex: • Find an eq. for the line parallel to $\vec{v} = \langle 1, 2, 3 \rangle$ through $P_0(1, 0, -1)$

$$\vec{r} = \vec{r}_0 + t\vec{v} = \vec{i} - \vec{k} + t(\vec{i} + 2\vec{j} + 3\vec{k}) = (1+t)\vec{i} + 2t\vec{j} + (3t-1)\vec{k}$$

– vector eq.

or $\quad x = 1+t, \quad y = 2t, \quad z = -1+3t$

– param. eq.

• Find two points on L other then P_0.

$t = 1 \qquad (2, 2, 2)$

$t = -1 \qquad (0, -2, -4)$

⚠ Vector and parametric equations of L are NOT unique!

e.g. we could take $P_0(2,2,2)$ $\rightsquigarrow$ $x=2+t$, $y=2+2t$, $z=2+3t$ – same line

another point on L

or could take $\langle 2,4,6\rangle$ as a direction vector

$\rightsquigarrow$ $x=1+2t$, $y=4t$, $z=-1+6t$ – same line

- Components of $\vec{v}=\langle a,b,c\rangle$ are called direction numbers of L.

Rmk: Any triple proportional to a,b,c can be used.

- Eliminate the parameter t from parametric equations:

$$t=\frac{x-x_0}{a} \qquad t=\frac{y-y_0}{b} \qquad t=\frac{z-z_0}{c} \implies$$

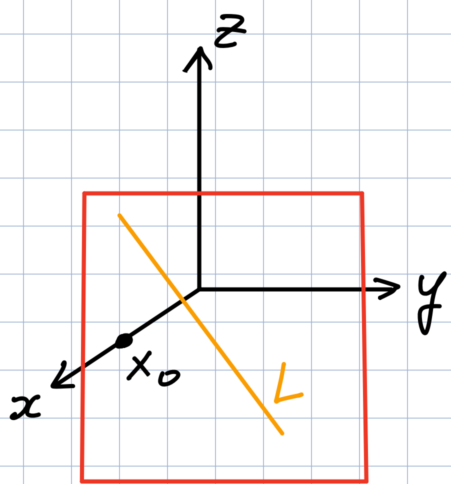
$$(t=) \quad \boxed{\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}}$$

symmetric equations of L

Q: What if one of a, b or c is 0?

A: If $a=0$ then

$$x = x_0 \ , \quad \frac{y-y_0}{b} = \frac{z-z_0}{c}$$



this means that L lies in the vertical plane $x=x_0$

Ex: $A=(1,0,2)$ $B(2,3,4)$

a) Find param. & sym. eq of the line L through A, B

b) Where does L intersect xy-plane?

a) $\vec{v} = \overrightarrow{AB} = \langle 1, 3, 2 \rangle$

$P_0 = A \rightsquigarrow \left.\begin{array}{l} x = 1+t \\ y = 3t \\ z = 2+2t \end{array}\right\}$ param. eq.

$\frac{x-1}{1} = \frac{y}{3} = \frac{z-2}{2}$ - sym. eq.

b) xy-plane: $z=0$ from $\frac{x-1}{1} = \frac{y}{3} = \frac{-2}{2} = -1 \Rightarrow y=-3, \ x=0$

$\left\{\begin{array}{l} \frac{x-1}{1} = \frac{-2}{2} \\ \frac{y}{3} = \frac{-2}{2} \end{array}\right.$

intersection point $(0,-3,0)$

• Line through $P_0(x_0, y_0, z_0)$, $P_1(x_1, y_1, z_1)$
has direction numbers $x_1 - x_0, y_1 - y_0, z_1 - z_0$

$\Rightarrow$ sym. eq. $\quad \dfrac{x - x_0}{x_1 - x_0} = \dfrac{y - y_0}{y_1 - y_0} = \dfrac{z - z_0}{z_1 - z_0}$

Ex: line segment from $A(1, 0, 2)$ to $B(2, 3, 4)$ is given by
$x = 1 + t,\ y = 3t,\ z = 2 + 2t$ with $0 \le t \le 1$
or by the corresponding vector equation
$r(t) = \langle 1 + t, 3t, 2 + 2t \rangle$ with $0 \le t \le 1$

• In general, line segment from $\vec{r_0}$ to $\vec{r_1}$ is given by

$$\vec{r} = \vec{r_0} + t(\vec{r_1} - \vec{r_0}) = (1 - t)\vec{r_0} + t\vec{r_1} \quad \text{with } 0 \le t \le 1$$

Ex: $L_1: x = 1+t,\ y = -2+3t,\ z = 4-t$ Show that this two lines are skew

$L_2: x = 2s,\ y = 3+s,\ z = -3+4s$ (i.e. they do not intersect and are not parallel ($\Rightarrow$ not lie in the same plane))

1) dir. vectors

$$\left.\begin{array}{l} v_1 = \langle 1, 3, -1 \rangle \\ v_2 = \langle 2, 1, 4 \rangle \end{array}\right\} \begin{array}{l} \text{are not parallel} \\ \Rightarrow L_1, L_2 \text{ not parallel} \end{array}$$

2) look for an intersection point: need (t, s) such that

$$\begin{array}{l} (1) \\ (2) \\ (3) \end{array} \begin{cases} 1+t = 2s \\ -2+3t = 3+s \\ 4-t = -3+4s \end{cases}$$

$(1)-2(2)$: $5-5t = -6 \Rightarrow t = \frac{11}{5}$

(1): $1+\frac{11}{5} = 2s \Rightarrow s = \frac{8}{5}$

check (3): $\underbrace{4-\frac{11}{5}}_{\frac{9}{5}} \neq -3 + \frac{32}{5}$

$\frac{9}{5} \neq \frac{17}{5}$

$\Rightarrow$ equations (1), (2), (3) are incompatible $\Rightarrow$ no intersection.

So L_1, L_2 are skew